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Expressing intrinsic volumes as rotational integrals

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ABSTRACT

A new rotational formula of Crofton type is derived for intrinsic volumes of a compact subset $X \subset \mathbb{R}^d$ of positive reach. The formula provides a functional defined on the section of X with a j-dimensional linear subspace with rotational average equal to the intrinsic volumes of X. Simplified forms of the functional are derived in special cases.

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1. Introduction

For a compact subset X of $\mathbb{R}^d$, satisfying certain regularity conditions, the classical Crofton formula relates integrals of intrinsic volumes defined on j-dimensional affine subspaces to intrinsic volumes of X,

$$\int_{\mathcal{F}_j^d} V_k(X \cap F_j)\,\mathrm{d}F_j^d = c_{d,j,k} V_{d-j+k}(X),$$

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$j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$. Here, $\mathcal{F}_j^d$ is the set of j-dimensional affine subspaces and $\mathrm{d}F_j^d$ is the element of the motion invariant measure on j-dimensional affine subspaces in $\mathbb{R}^d$. Furthermore, $V_k(X)$, $k = 0, 1, \ldots, d$, are the intrinsic volumes of X. Finally, $c_{d,j,k}$ is a known constant.

Motivated by applications in local stereology [2], a rotational version of the Crofton formula has recently been derived, cf. [8]. This formula shows how rotational averages of intrinsic volumes measured on sections passing through a fixed point are related to the geometry of the sectioned object. More specifically, for a compact subset $X \subset \mathbb{R}^d$ of positive reach, the functionals $\beta_{j,k}$, satisfying

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\,\mathrm{d}L_j^d = \beta_{j,k}(X),$$

$j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$, have been determined in [8]. For $k = j$, $\beta_{j,j}(X)$ is a simple integral while in the case $k < j$, $\beta_{j,k}(X)$ is a complicated integral over the unit normal bundle of X, involving principal curvatures and hypergeometric functions.

In the present paper, we address the 'opposite' problem of finding functionals $\alpha_{j,k}$, satisfying the following rotational integral equation

$$\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j)\,\mathrm{d}L_j^d = V_{d-j+k}(X), \tag{1}$$

$j = 0, 1, \ldots, d$ and $k = 0, 1, \ldots, j$. The solution of the problem is inspired by some recent work reported in [3] and [4].

2. The general solution

The main tools for deriving solutions to (1) are the classical Crofton formula and a well-known geometric measure decomposition from integral geometry.

The motion invariant measure on j-dimensional affine subspaces can be decomposed as follows. For $F_j = x + L_j$, where L_j is a j-dimensional linear subspace and $x \in L_j^\perp$, we have $\mathrm{d}F_j^d = \mathrm{d}x^{d-j}\,\mathrm{d}L_j^d$ where $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$, the set of j-dimensional linear subspaces and, for given $L_j \in \mathcal{L}_j^d$, $\mathrm{d}x^{d-j}$ is the element of the Lebesgue measure in $L_j^\perp$. The total mass of $\mathrm{d}L_j^d$ is chosen to be

$$\int_{\mathcal{L}_j^d} \mathrm{d}L_j^d = c_{d,j},$$

where

$$c_{d,j} = \frac{\sigma_d \sigma_{d-1} \cdots \sigma_{d-j+1}}{\sigma_j \sigma_{j-1} \cdots \sigma_1} \tag{2}$$

and $\sigma_k = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of the unit sphere in $\mathbb{R}^k$. With this choice, the constant in the classical Crofton formula becomes

$$c_{d,j,k} = c_{d,j} \cdot \frac{\Gamma(\frac{j+1}{2})\Gamma(\frac{d+k-j+1}{2})}{\Gamma(\frac{k+1}{2})\Gamma(\frac{d+1}{2})}. \tag{3}$$

The geometric measure decomposition used in the derivation of solutions to (1) concerns the motion invariant measure on r-dimensional affine subspaces in $\mathbb{R}^d$. According to Gual-Arnau and Cruz-Orive [4], we have for $r = 0, 1, \dots, d-1$ that

$$\mathrm{d}F_r^d = d(O, F_r)^{d-r-1}\, \mathrm{d}F_r^{r+1}\, \mathrm{d}L_{r+1}^d, \tag{4}$$

where $\mathrm{d}F_r^{r+1}$ is the element of the motion invariant measure on r-dimensional affine subspaces in L_{r+1} and $d(O, F_r)$ denotes the distance from F_r to the origin O. Note that for $r = 0$, (4) reduces to the standard polar decomposition of Lebesgue measure

$$\mathrm{d}x^d = |x|^{d-1}\, \mathrm{d}x^1\, \mathrm{d}L_1^d.$$

We formulate the main result of this paper in the proposition below.

Proposition 1. *Let X be a compact subset of $\mathbb{R}^d$ of positive reach. Assume that for almost all $L_j \in \mathcal{L}_j^d$*

$$(x, n) \in \operatorname{nor} X, \quad x \in L_j \quad \Rightarrow \quad n \not\perp L_j, \tag{5}$$

where nor X *is the unit normal bundle of X. Then,*

$$\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j)\, \mathrm{d}L_j^d = V_{d-j+k}(X),$$

$j = 1, \dots, d$, $k = 1, \dots, j$, where

$$\alpha_{j,k}(X \cap L_j) = \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j}\, V_{k-1}\big((X \cap L_j) \cap F_{j-1}\big)\, \mathrm{d}F_{j-1}^j. \tag{6}$$

Proof. The condition (5) of the proposition ensures that $X \cap L_j$ is of positive reach for almost all $L_j \in \mathcal{L}_j^d$, cf. [8, p. 550]. Using the Crofton formula and the measure decomposition (4), we find that

$$\begin{aligned}
\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j)\, \mathrm{d}L_j^d &= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}_j^d} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j}\, V_{k-1}(X \cap L_j \cap F_{j-1})\, \mathrm{d}F_{j-1}^j\, \mathrm{d}L_j^d \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}_j^d} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-(j-1)-1}\, V_{k-1}(X \cap F_{j-1})\, \mathrm{d}F_{j-1}^j\, \mathrm{d}L_j^d \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}_{j-1}^d} V_{k-1}(X \cap F_{j-1})\, \mathrm{d}F_{j-1}^d \\
&= V_{d-j+k}(X). \qquad \square
\end{aligned}$$

3. The case $k = j$

For $k = j$, Proposition 1 provides a functional with rotational average equal to the volume $V_d(X)$. This functional can be simplified considerably, as shown in the proposition below. We use here and in the following the notation $p(x|L_r)$ for the orthogonal projection of $x \in \mathbb{R}^d$ onto $L_r \in \mathcal{L}_r^d$.

Proposition 2. *Let the situation be as in Proposition* 1 *and suppose that $k = j$. Then,*

$$\alpha_{j,j}(X \cap L_j) = \frac{1}{c_{d-1,j-1}} \int_{X \cap L_j} |z|^{d-j} \, \mathrm{d}z^j.$$

Proof. Using that $F_{j-1} = L_{j-1} + x$, where $x \in L_{j-1}^{\perp}$, we find

$$\begin{aligned}
\alpha_{j,j}(Y) &= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j} V_{j-1}(Y \cap F_{j-1}) \, \mathrm{d}F_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^{\perp}} |x|^{d-j} V_{j-1}\big(Y \cap (L_{j-1} + x)\big) \, \mathrm{d}x^1 \, \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^{\perp}} \int_{Y \cap (L_{j-1}+x)} |x|^{d-j} \, \mathrm{d}y^{j-1} \, \mathrm{d}x^1 \, \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{Y} \big|p\big(z|L_{j-1}^{\perp}\big)\big|^{d-j} \, \mathrm{d}z^j \, \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{Y} |z|^{d-j} \left(\int_{\mathcal{L}_{j-1}^j} \frac{|p(z|L_{j-1}^{\perp})|^{d-j}}{|z|^{d-j}} \, \mathrm{d}L_{j-1}^j \right) \mathrm{d}z^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{Y} |z|^{d-j} \left(\frac{c_{j,j-1}}{B(\frac{1}{2}, \frac{j-1}{2})} \int_0^1 y^{\frac{d-j-1}{2}} (1-y)^{\frac{j-3}{2}} \, \mathrm{d}y \right) \mathrm{d}z^j.
\end{aligned}$$

At the last equality sign, we have used [7, Proposition 3.9]. The result now follows immediately, using (2) and (3). □

4. The case $k < j$

It is also possible to make the expression of the functional $\alpha_{j,k}$ more explicit for $k < j$. We will concentrate on the case where X is the compact closure of an open subset of $\mathbb{R}^d$ and ∂X is a $(d-1)$-dimensional manifold of class C^2. For $k = 0, 1, \ldots, d-1$, the kth intrinsic volume has the following integral representation

$$V_k(X) = \frac{1}{\sigma_{d-k}} \int_{\partial X} \sum_{|I|=d-1-k} \prod_{i \in I} \kappa_i(x) \, \mathcal{H}^{d-1}(\mathrm{d}x), \tag{7}$$

where $\kappa_i(x)$, $i = 1, \dots, d-1$, are the principal curvatures of ∂X at $x \in \partial X$ and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. We will assume that for all $j = 1, \dots, d$, almost all $z \in \partial X$ and almost all $L_j \in \mathcal{L}_j^d$

$$x \in (\partial X) \cap (L_j + z) \quad \Rightarrow \quad n(x) \not\perp L_j. \tag{8}$$

For an affine subspace $F_j = L_j + z$, satisfying (8), we have, cf. [5, pp. 59 and 60],

$$\partial(X \cap F_j) = (\partial X) \cap F_j$$

and $\partial(X \cap F_j)$ is a $(j-1)$-dimensional manifold of class C^2. The principal curvatures of $\partial(X \cap F_j)$ at $x \in \partial(X \cap F_j)$ are denoted by $\kappa_{F_j,i}(x)$, $i = 1, \dots, j-1$.

The proposition below gives a more explicit expression for $\alpha_{j,k}$ for $k < j$ than the one given in (6).

Proposition 3. *Let the situation be as in Proposition 1 and let $k < j$. Suppose that X is the compact closure of an open subset of $\mathbb{R}^d$ and ∂X is a $(d-1)$-dimensional manifold of class C^2 for which (8) is satisfied. Then,*

$$\begin{aligned} &c_{d,j-1,k-1}\sigma_{j-k}\alpha_{j,k}(X \cap L_j) \\ &\quad = \int\limits_{\partial(X\cap L_j)} \int\limits_{\mathcal{L}_{j-1}^j} \kappa(z; L_{j-1}+z)\big|p\big(n(z)|L_{j-1}\big)\big|\big|p\big(z|L_{j-1}^\perp\big)\big|^{d-j}\, \mathrm{d}L_{j-1}^j\, \mathcal{H}^{j-1}(\mathrm{d}z), \end{aligned}$$

where $n(z)$ is the unit normal of $\partial(X \cap L_j)$ at z and

$$\kappa(z; F_{j-1}) = \begin{cases} 1 & \text{if } k = j-1, \\ \sum_{|I|=j-k-1} \prod_{i\in I} \kappa_{F_{j-1},i}(z) & \text{if } k < j-1. \end{cases}$$

Proof. Note that the condition (8) ensures that for almost all $L_j \in \mathcal{L}_j^d$, $z \in \partial(X \cap L_j)$ and $L_{j-1} \in \mathcal{L}_{j-1}^j$, $\partial(X \cap L_j) \cap (L_{j-1} + z)$ is a $(j-2)$-dimensional manifold of class C^2. This can be seen by first noting that

$$\begin{aligned} \partial(X \cap L_j) \cap (L_{j-1}+z) &= (\partial X) \cap L_j \cap (L_{j-1}+z) \\ &= (\partial X) \cap (L_{j-1}+z), \end{aligned}$$

and then combining (8) with [7, Proposition 5.4]. The function $\kappa(z; L_{j-1}+z)$ is well defined when $\partial(X \cap L_j) \cap (L_{j-1}+z)$ is a $(j-2)$-dimensional manifold of class C^2.

Letting $Y = X \cap L_j$, we have according to (6)

$$\begin{aligned} \alpha_{j,k}(Y) &= \frac{1}{c_{d,j-1,k-1}} \int\limits_{\mathcal{F}_{j-1}^j} \mathrm{d}(O, F_{j-1})^{d-j}\, V_{k-1}(Y \cap F_{j-1})\, \mathrm{d}F_{j-1}^j \\ &= \frac{1}{c_{d,j-1,k-1}} \int\limits_{\mathcal{L}_{j-1}^j} \int\limits_{L_{j-1}^\perp} |x|^{d-j} V_{k-1}\big(Y \cap (L_{j-1}+x)\big)\, \mathrm{d}x^1\, \mathrm{d}L_{j-1}^j. \end{aligned}$$

Using the integral representation (7) of intrinsic volumes, the expression above becomes

$$\begin{aligned}
&c_{d,j-1,k-1}\alpha_{j,k}(Y) \\
&= \frac{1}{\sigma_{(j-1)-(k-1)}} \int_{\mathcal{L}^j_{j-1}} \int_{L^{\perp}_{j-1}} |x|^{d-j} \int_{\partial Y \cap (L_{j-1}+x)} \kappa(y; L_{j-1}+x)\, \mathcal{H}^{(j-1)-1}(\mathrm{d}y)\, \mathrm{d}x^1\, \mathrm{d}L^j_{j-1} \\
&= \frac{1}{\sigma_{j-k}} \int_{\mathcal{L}^j_{j-1}} \int_{L^{\perp}_{j-1}} \int_{\partial Y \cap (L_{j-1}+x)} \left|p\left(y|L^{\perp}_{j-1}\right)\right|^{d-j} \kappa(y; L_{j-1}+y)\, \mathcal{H}^{j-2}(\mathrm{d}y)\, \mathrm{d}x^1\, \mathrm{d}L^j_{j-1}.
\end{aligned}$$

At the first equality sign we have used that $\partial(Y \cap F_{j-1}) = \partial Y \cap F_{j-1}$ for almost all F_{j-1}. Using [7, Propositions 2.10 and 5.2] and Fubini, we finally get

$$\begin{aligned}
&c_{d,j-1,k-1}\alpha_{j,k}(Y) \\
&= \frac{1}{\sigma_{j-k}} \int_{\mathcal{L}^j_{j-1}} \int_{\partial Y} \left|p\left(n(z)|L_{j-1}\right)\right| \left|p\left(z|L^{\perp}_{j-1}\right)\right|^{d-j} \kappa(z, L_{j-1}+z)\, \mathcal{H}^{j-1}(\mathrm{d}z)\, \mathrm{d}L^j_{j-1} \\
&= \frac{1}{\sigma_{j-k}} \int_{\partial Y} \int_{\mathcal{L}^j_{j-1}} \kappa(z, L_{j-1}+z) \left|p\left(n(z)|L_{j-1}\right)\right| \left|p\left(z|L^{\perp}_{j-1}\right)\right|^{d-j} \mathrm{d}L^j_{j-1}\, \mathcal{H}^{j-1}(\mathrm{d}z). \qquad \square
\end{aligned}$$

For $k = j-1$, the expression for $\alpha_{j,k}(Y)$ given in Proposition 3 can be further simplified, using the following proposition. The proof is deferred to Appendix A.

Proposition 4. *Let $L_j \in \mathcal{L}^d_j$, $j = 1, \ldots, d$. Let x and y be unit vectors in L_j. Then, for all $m, n \in \mathbb{N}$,*

$$\begin{aligned}
&\int_{\mathcal{L}^j_{j-1}} \left|p(x|L_{j-1})\right|^m \left|p\left(y|L^{\perp}_{j-1}\right)\right|^n \mathrm{d}L^j_{j-1} \\
&\quad = \frac{\sigma_{j-1}}{2} B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{j-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{j-1}{2}, \sin^2 \angle(x, y)\right). \qquad \square
\end{aligned}$$

Using Proposition 4 with $m = 1$ and $n = d-j$, we find

$$\alpha_{j,j-1}(Y) = \frac{1}{2c_{d-1,j-1}} \int_{\partial Y} |z|^{d-j} F\left(-\frac{1}{2}, -\frac{d-j}{2}; \frac{j-1}{2}; \sin^2 \angle\left(n(z), z\right)\right) \mathcal{H}^{j-1}(\mathrm{d}z).$$

Appendix A

In this appendix, we will prove Proposition 4. Without loss of generality, we assume that $x \cdot y > 0$. For simplicity, we write $\mathrm{d}z^j$ instead of $\mathcal{H}^j(\mathrm{d}z)$.

The Gauss *hypergeometric series* or *hypergeometric function* is defined for $a, b, c \in \mathbb{R}$ and $z \in [-1, 1]$ as

$$F(a, b; c; z) = F(b, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(x)_k$ is the *rising sequential product* or *Pochhammer symbol* defined for a non-negative integer k and $x \in \mathbb{R}$ by

$$(x)_k = \begin{cases} \frac{\Gamma(x+k)}{\Gamma(x)} & \text{if } x > 0, \\ (-1)^k \frac{\Gamma(-x+1)}{\Gamma(-x-k+1)} & \text{if } x \leqslant 0. \end{cases}$$

Note that $(x)_k = 0$ whenever $x \in \{0, -1, -2, \ldots\}$ and $k > -x$.

An application of [7, Propositions 3.2 and 3.3] gives

$$\begin{aligned} \int_{\mathcal{L}^j_{j-1}} \left|p(x|L_{j-1})\right|^m \left|p\left(y|L^\perp_{j-1}\right)\right|^n \mathrm{d}L^j_{j-1} &= \int_{\mathcal{L}^j_1} \left|p\left(x|L^\perp_1\right)\right|^m \left|p(y|L_1)\right|^n \mathrm{d}L^j_1 \\ &= \frac{1}{2} \int_{S^{j-1}} \left|p\left(x|\mathrm{span}\{\omega\}^\perp\right)\right|^m \left|p\left(y|\mathrm{span}\{\omega\}\right)\right|^n \mathrm{d}\omega^{j-1} \\ &= \frac{1}{2} \int_{S^{j-1}} \sqrt{1-(x\cdot\omega)^2}^{\,m} |y\cdot\omega|^n \,\mathrm{d}\omega^{j-1} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \binom{\frac{m}{2}}{k} (-1)^k \int_{S^{j-1}} |x\cdot\omega|^{2k} |y\cdot\omega|^n \,\mathrm{d}\omega^{j-1}. \end{aligned} \tag{9}$$

Now note that

$$\int_{S^{j-1}} |x\cdot\omega|^{2k} |y\cdot\omega|^n \,\mathrm{d}\omega^{j-1} = \int_{S^{j-1}} \left|p\left(p(\omega|x\oplus y)|x\right)\right|^{2k} \left|p\left(p(\omega|x\oplus y)|y\right)\right|^n \mathrm{d}\omega^{j-1}. \tag{10}$$

In order to compute (10), we will use the following lemma.

Lemma 1. *Let $B_p \in \mathcal{L}^d_p$. Then, for any non-negative measurable function $g : \mathbb{R}^d \to \mathbb{R}$,*

$$\int_{S^{d-1}} g\left(p(x|B_p)\right) \mathrm{d}x^{d-1} = \frac{\sigma_{d-p}}{2} \int_{S^{p-1}(B_p)} \int_0^1 g\left(t^{\frac{1}{2}} x_0\right) t^{\frac{p-2}{2}} (1-t)^{\frac{d-p-2}{2}} \,\mathrm{d}t \,\mathrm{d}x_0^{p-1},$$

where $S^{p-1}(B_p)$ is the unit sphere in B_p.

Proof. First, we use the co-area formula with

$$\begin{gathered} \psi : S^{d-1} \setminus B^\perp_p \to S^{p-1}(B_p), \\ x \to \pi(x|B_p) := p(x|B_p)/\left|p(x|B_p)\right|. \end{gathered}$$

The $(p-1)$-dimensional Jacobian of ψ is given by

$$J_{p-1}\psi\left(x, S^{d-1}\right) = \left|p(x|B_p)\right|^{-(p-1)}.$$

Hence, the co-area formula yields

$$\begin{aligned}\int\limits_{S^{d-1}} g\big(p(x|B_p)\big)\,\mathrm{d}x^{d-1} &= \int\limits_{S^{d-1}} g\big(\big|p(x|B_p)\big|\pi(x|B_p)\big)\,\mathrm{d}x^{d-1} \\ &= \int\limits_{S^{p-1}(B_p)}\ \int\limits_{\psi^{-1}\{x_0\}} g\big(\big|p(x|B_p)\big|x_0\big)\big|p(x|B_p)\big|^{p-1}\,\mathrm{d}x^{d-p}\,\mathrm{d}x_0^{p-1}. \qquad (11)\end{aligned}$$

Next, let $x_0 \in S^{p-1}(B_p)$ be fixed and apply the area formula with

$$\xi : B_p^{\perp} \to \psi^{-1}\{x_0\},$$
$$\omega \mapsto \frac{\omega + x_0}{|\omega + x_0|}.$$

The $(d-p)$-dimensional Jacobian of ξ is

$$J_{d-p}\xi\big(\omega, B_p^{\perp}\big) = \left(\frac{1}{1+|\omega|^2}\right)^{\frac{d-p+1}{2}}.$$

Hence, since ξ maps $B_p^{\perp}$ bijectively onto $\psi^{-1}\{x_0\}$ and $|p(\xi(\omega)|B_p)| = \frac{1}{|\omega+x_0|} = (\frac{1}{1+|\omega|^2})^{\frac{1}{2}}$, we have

$$\begin{aligned}&\int\limits_{\psi^{-1}\{x_0\}} g\big(\big|p(x|B_p)\big|x_0\big)\big|p(x|B_p)\big|^{p-1}\,\mathrm{d}x^{d-p} \\ &\quad= \int\limits_{\psi^{-1}\{x_0\}} g\big(\big|p(x|B_p)\big|x_0\big)\big|p(x|B_p)\big|^{p-1}\,\mathrm{d}x^{d-p} \\ &\quad= \int\limits_{\psi^{-1}\{x_0\}} g\left(\left(\frac{1}{1+|\xi^{-1}(x)|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|\xi^{-1}(x)|^2}\right)^{\frac{p-1}{2}}\,\mathrm{d}x^{d-p} \\ &\quad= \int\limits_{B_p^{\perp}} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{p-1}{2}}\left(\frac{1}{1+|x|^2}\right)^{\frac{d-p+1}{2}}\,\mathrm{d}x^{d-p} \\ &\quad= \int\limits_{B_p^{\perp}} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{d}{2}}\,\mathrm{d}x^{d-p}.\end{aligned}$$

Using [7, Proposition 2.8], we get

$$\begin{aligned}&\int\limits_{B_p^{\perp}} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{d}{2}}\,\mathrm{d}x^{d-p} \\ &\quad= \sigma_{d-p}\int\limits_0^{\infty} g\left(\left(\frac{1}{1+t^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+t^2}\right)^{\frac{d}{2}} t^{d-p-1}\,\mathrm{d}t. \qquad (12)\end{aligned}$$

Substitution with $s = \frac{1}{1+t^2}$ yields

$$\int_0^\infty g\biggl(\biggl(\frac{1}{1+t^2}\biggr)^{\frac{1}{2}} x_0\biggr)\biggl(\frac{1}{1+t^2}\biggr)^{\frac{d}{2}} t^{d-p-1}\,\mathrm{d}t = \frac{1}{2}\int_0^1 g\bigl(s^{\frac{1}{2}} x_0\bigr) s^{\frac{p-2}{2}}(1-s)^{\frac{d-p-2}{2}}\,\mathrm{d}s.$$

The last equation combined with (11) and (12) implies

$$\int_{S^{d-1}} g\bigl(p(x|B_p)\bigr)\,\mathrm{d}x^{d-1} = \frac{\sigma_{d-p}}{2}\int_{S^{p-1}(B_p)}\int_0^1 g\bigl(t^{\frac{1}{2}} x_0\bigr) t^{\frac{p-2}{2}}(1-t)^{\frac{d-p-2}{2}}\,\mathrm{d}t\,\mathrm{d}x_0^{p-1}. \qquad \square$$

Applying Lemma 1 with $B = \operatorname{span}\{x, y\}$, we get

$$\begin{aligned}
&\int_{S^{j-1}} \bigl|p\bigl(p(\omega|x\oplus y)|x\bigr)\bigr|^{2k}\bigl|p\bigl(p(\omega|x\oplus y)|y\bigr)\bigr|^{n}\,\mathrm{d}\omega^{j-1} \\
&\quad = \frac{\sigma_{j-2}}{2}\int_{S^1(B)}\int_0^1 t^k\bigl|p(\omega_0|x)\bigr|^{2k} t^{n/2}\bigl|p(\omega_0|y)\bigr|^n t^{\frac{2-2}{2}}(1-t)^{\frac{j-2-2}{2}}\,\mathrm{d}t\,\mathrm{d}\omega_0^1 \\
&\quad = \frac{\sigma_{j-2}}{2}\int_{S^1(B)} \bigl|p(\omega_0|y)\bigr|^n\bigl|p(\omega_0|x)\bigr|^{2k}\,\mathrm{d}\omega_0^1\int_0^1 t^{\frac{n+2k}{2}}(1-t)^{\frac{j-4}{2}}\,\mathrm{d}t \\
&\quad = \frac{\sigma_{j-2}B(\frac{n}{2}+k+1, \frac{j-2}{2})}{2}\int_{S^1(B)} \bigl|p(\omega_0|y)\bigr|^n\bigl|p(\omega_0|x)\bigr|^{2k}\,\mathrm{d}\omega_0^1. \qquad (13)
\end{aligned}$$

Successive application of [7, Proposition 3.2] and [6, Corollary 4.2] yields

$$\begin{aligned}
&\int_{S^1(B)} \bigl|p(\omega_0|y)\bigr|^n\bigl|p(\omega_0|x)\bigr|^{2k}\,\mathrm{d}\omega_0^1 = 2\int_{\mathcal{L}_1^2(B)} \bigl|p(x|L_1)\bigr|^{2k}\bigl|p(y|L_1)\bigr|^n\,\mathrm{d}L_1^2 \\
&= 2\int_{-1}^1\int_{S^1(B)\cap y^\perp} \bigl(1-t^2\bigr)^{\frac{2-1-2}{2}}\bigl|p\bigl(x|ty+\sqrt{1-t^2}\omega\bigr)\bigr|^{2k}\bigl|p\bigl(y|ty+\sqrt{1-t^2}\omega\bigr)\bigr|^n\,\mathrm{d}\omega\,\mathrm{d}t \\
&= 2\int_{-1}^1\int_{S^1(B)\cap y^\perp} \bigl(1-t^2\bigr)^{\frac{2-1-2}{2}}|t|^n\bigl|t(y\cdot x)+\sqrt{1-t^2}(x\cdot\omega)\bigr|^{2k}\,\mathrm{d}\omega\,\mathrm{d}t \\
&= 2\int_{-1}^1 \bigl(1-t^2\bigr)^{\frac{2-1-2}{2}}|t|^n\Bigl(\bigl|t(y\cdot x)+\sqrt{1-t^2}\sqrt{1-(y\cdot x)^2}\bigr|^{2k} \\
&\quad + \bigl|t(y\cdot x)-\sqrt{1-t^2}\sqrt{1-(y\cdot x)^2}\bigr|^{2k}\Bigr)\,\mathrm{d}t.
\end{aligned}$$

Using the binomial formula, the last expression becomes

$$2\sum_{l=0}^{2k}\binom{2k}{l}\int_{-1}^{1}\Big(\big(1-t^2\big)^{\frac{2-1-2}{2}}|t|^n t^l (y\cdot x)^l\big(1-t^2\big)^{\frac{2k-l}{2}}\sqrt{1-(x\cdot y)^2}^{\,2k-l}$$

$$+(-1)^l\big(1-t^2\big)^{\frac{2-1-2}{2}}|t|^n t^l (y\cdot x)^l\big(1-t^2\big)^{\frac{2k-l}{2}}\sqrt{1-(x\cdot y)^2}^{\,2k-l}\Big)\,\mathrm{d}t$$

$$=4\sum_{l=0}^{k}(x\cdot y)^{2l}\big(1-(x\cdot y)^2\big)^{k-l}\binom{2k}{2l}\int_{0}^{1}\big(1-t^2\big)^{k-l-\frac{1}{2}}t^{n+2l}\,\mathrm{d}t$$

$$=2\sum_{l=0}^{k}(x\cdot y)^{2l}\big(1-(x\cdot y)^2\big)^{k-l}\binom{2k}{2l}B\left(\frac{n}{2}+l+\frac{1}{2},k-l+\frac{1}{2}\right).$$

Applying the duplication formula for the Gamma function,

$$\Gamma(2z)=\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)\pi^{-\frac{1}{2}}2^{2z-1},$$

we obtain

$$2\sin^{2k}\angle(x,y)B\left(\frac{n}{2}+\frac{1}{2},k+\frac{1}{2}\right)\sum_{l=0}^{k}\frac{(-k)_l(\frac{n}{2}+\frac{1}{2})_l}{(\frac{1}{2})_l}\frac{(-\tan^{-2}\angle(x,y))^l}{l!}$$

$$=2\sin^{2k}\angle(x,y)B\left(\frac{n}{2}+\frac{1}{2},k+\frac{1}{2}\right)F\left(-k,\frac{n}{2}+\frac{1}{2};\frac{1}{2};-\tan^{-2}\angle(x,y)\right).$$

According to [1, (15.3.4)] with $z=\cos^2\angle(x,y)$,

$$\sin^{2k}\angle(x,y)F\left(-k,\frac{n}{2}+\frac{1}{2};\frac{1}{2};-\tan^{-2}\angle(x,y)\right)=F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right).$$

By insertion in (13), we get

$$\int_{S^{j-1}}|x\cdot\omega|^{2k}|y\cdot\omega|^n\,\mathrm{d}\omega^{j-1}$$

$$=\sigma_{j-2}B\left(\frac{n}{2}+k+1,\frac{j-2}{2}\right)B\left(k+\frac{1}{2},\frac{n}{2}+\frac{1}{2}\right)F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right).$$

Hence, (9) becomes

$$\int_{\mathcal{L}^j_{j-1}}\big|p(x|L_{j-1})\big|^m\big|p\big(y|L^{\perp}_{j-1}\big)\big|^n\,\mathrm{d}L^j_{j-1}$$

$$=\frac{\sigma_{j-2}}{2}\sum_{k=0}^{\infty}\binom{\frac{m}{2}}{k}(-1)^k B\left(\frac{n}{2}+k+1,\frac{j-2}{2}\right)B\left(\frac{n+1}{2},k+\frac{1}{2}\right)F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right).$$

Since

$$\begin{aligned}&\frac{\sigma_{j-2}}{2}\binom{\frac{m}{2}}{k}(-1)^k B\left(\frac{n}{2}+k+1, \frac{j-2}{2}\right) B\left(\frac{n+1}{2}, k+\frac{1}{2}\right)\\ &\quad = \frac{\sigma_{j-1}}{2} B\left(\frac{j-1}{2}, \frac{n+1}{2}\right)\frac{(-\frac{m}{2})_k}{k!}\frac{(\frac{1}{2})_k}{(\frac{n+j}{2})_k},\end{aligned}$$

we now have

$$\begin{aligned}&\int\limits_{\mathcal{L}^j_{j-1}} \left|p(x|L_{j-1})\right|^m \left|p\left(y|L^{\perp}_{j-1}\right)\right|^n \,\mathrm{d}L^j_{j-1}\\ &\quad = \frac{\sigma_{j-1}}{2} B\left(\frac{j-1}{2}, \frac{n+1}{2}\right)\sum_{k=0}^{\infty}\frac{(-\frac{m}{2})_k(\frac{1}{2})_k}{(\frac{n+j}{2})_k}\frac{F(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2\angle(x, y))}{k!}.\end{aligned}$$

Using the power series expansion of the hypergeometric function, then expanding $(1-\sin^2\angle(x, y))^k$ and applying the identities

$$\frac{\binom{k+l}{l}}{(k+l)!} = \frac{1}{l!}\frac{1}{k!} \quad \text{and} \quad (a)_{k+l} = (a)_l(a+l)_k,$$

it is straightforward to prove that the last expression equals

$$\frac{\sigma_{j-1}}{2} B\left(\frac{n+1}{2}, \frac{m}{2}+\frac{j-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{j-1}{2}; \sin^2\angle(x, y)\right).$$

The proof is complete.

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